Mathematical Morphology

Erosions and Dilations

Luc Brun (d'après le cours de M. Coster)

25 janvier 2018

Plan (1/2)

Plan (2/2)

Basic idea of mathematical morphology

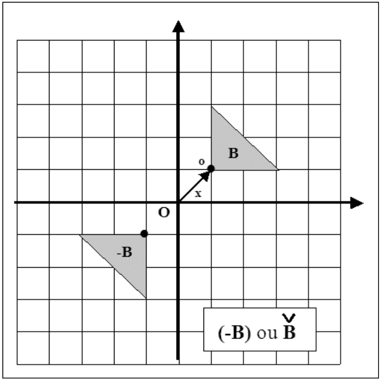
- Recall :

  The basic idea of mathematical morphology is to compare the set to be analysed with a set with a known geometry called structuring element.

- A struturing element $B$is a set with the following characteristics :
  - It has a known geometry
  - A size λ,
  - This element is located by its origin o. The origin o usually belongs to the structuring element but this is not mandatory.
- Examples :

Transpose of a structuring element

- Definition :
  The transpose of a struturing element B (denoted $\check{B}$ or $-B$) is the structuring element mirrored in the origin o.



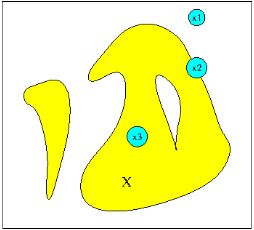
Set Erosion and Dilatation

- Struncturing element
  - Définition,
  - Example,
  - Transpose
- colorred Set erosions and dilatations
  - Hit Miss Transformation
  - Erosion
  - Example of Erosion
  - Bi colored transformation
  - Erosion and Minkowski's substraction
  - Dilatation
  - Dilatation : Examples
  - Dilatation and Minkowski's addition
- Properties of set erosion and dilatation
  - Duality,
  - Extensiveness
  - Growth
  - Composition
  - Union, Intersection
  - Composition
  - Above continuity of the erosion
- Distance computations
  - Distance of a point to a set
  - Distance and slices
  - Distance by erosion : Algorithm
  - Examples
  - External Distance

Binary transform

- A binary transformation of X by B in E ,is defined by moving B on the whole set of points $x \in E$. For each position, we ask a question relative to the union, the intersection or the inclusion of B with X. The set of points corresponding to a positive answer defines the transformed image.
- Simplest binary transformations are :
  - The erosion which is a transformation relative to the inclusion.
  - The dilation which is relative to the intersection.

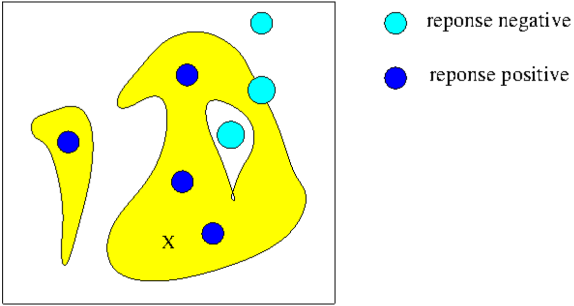
It exists more complex binary transformations like the hit or miss using two structuring elements.



Erosion

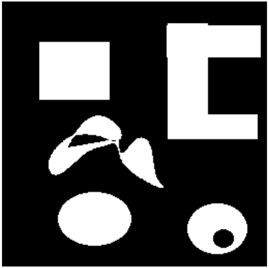
- Definition :
  The structuring element B, located by its origin is moved so as to occupy successively all positions of the space E. For each position we wonder does B is completely included in X ?
- The set of positive answers provide the eroded set.

$$\mathcal{E}_B(X) = \{x \in E, B_x \subset X\}$$

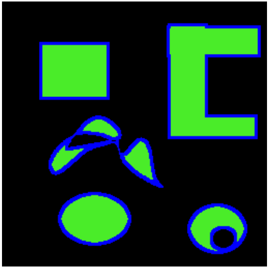


Erosion : Example (1)

- $B =$ ■ (radius 3)



X



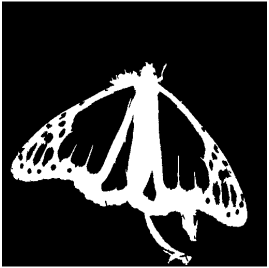
X(■+■) et $\mathcal{E}_B(X)$(■)

$\mathcal{E}_B(X)$

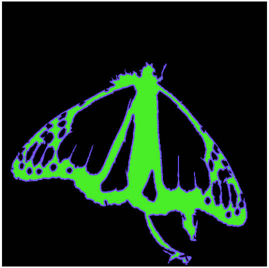
- Qualitative properties
  - The size of objects decreases
  - An object with holes or concavities may split into several connected components
  - Small objects and details disappear

Erosion : Example (2)

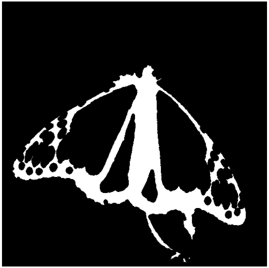
- $B = \bullet$ (radius 5)



X



$X(\blacksquare + \blacksquare)$ et $\mathcal{E}_B(X)(\blacksquare)$

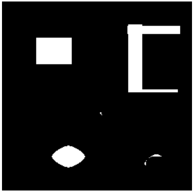


$\mathcal{E}_B(X)$

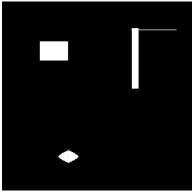
Erosion with structuring elements of increasing size

X

$\mathcal{E}_{5B}(X)$

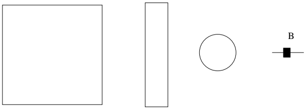


$\mathcal{E}_{10B}(X)$

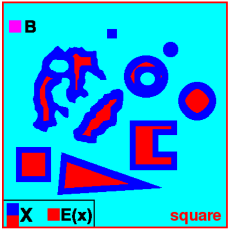


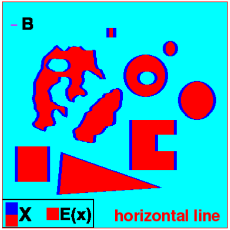
$\mathcal{E}_{15B}(X)$

- Quid of $\mathcal{E}_B(X)$?



- Erosion with different structuring elements





- The structuring element is composed of two subsets corresponding to two labels : $B = B^0 \cup B^1$.
- We say that x belongs to the Hit and Miss transform of X by B iff :

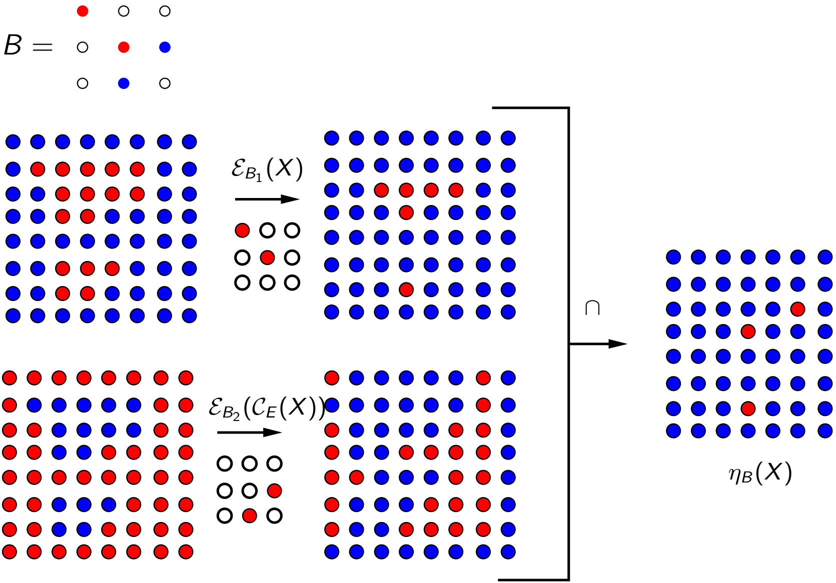
$$B_x^0 \subset \mathcal{C}_E(X) \text{ and } B_x^1 \subset X$$

- An Hit and Miss transform may be written as the intersection of two erosions on X and $\mathcal{C}_E(X)$.

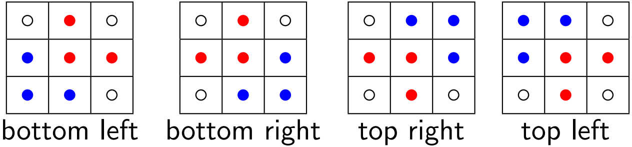
$$\eta_B(X) = \eta_{B^0,B^1}(X) = \mathcal{E}_{B^1}(X) \cap \mathcal{E}_{B^2}(\mathcal{C}_E(X))$$

Illustration

- $B =$



- Corner detection



Erosion and Minkowski's subtraction

- Set erosion is equivalent to Minkowski subtraction by the transposed element

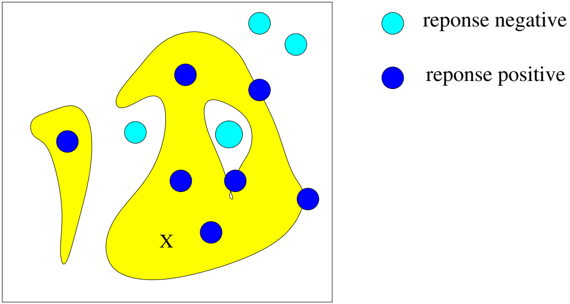
$$\mathcal{E}_B(X) = X \ominus \breve{B} = \bigcap_{b \in B} X_{-b}$$

- Demonstration : Let $z \in \bigcap_{b \in B} X_{-b} \ldots$(to do)

Set dilatation

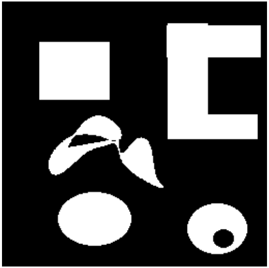
- Dilatation is a binary transformation based on the intersection.
- Definition :
  The structuring element B, located by its origin is moved on all positions of the space E. For each position we check does B intersects X ?
- The set of positive answers forms the dilated set.

$$\delta_B(X) = \{x \in E, B_x \cap X \neq \emptyset\}$$

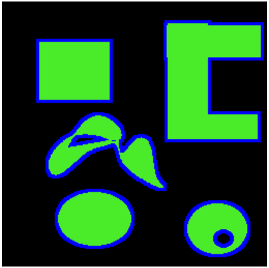


Dilatation : Example (1)

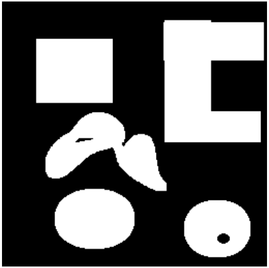
- $B =$ ■ (radius 3)



X



$\delta_B(X)$(■+■) et$X$(■)

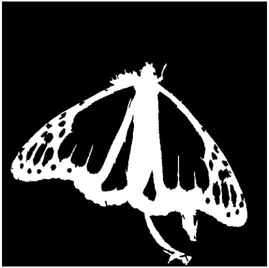


$\delta_B(X)$

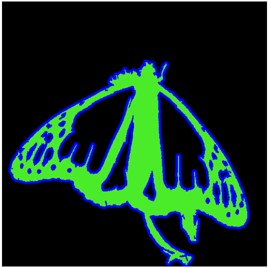
- Qualitative properties
  - The size of objects increases
  - Holes and concavities may be filled
  - Close objects may become connected
  - Small details disappear

Dilatation : Example (2)

- $B = \bullet$ (radius 5)



X



$\delta_B(X)$(■+■) et$X$(■)

$\delta_B(X)$

Dilatation with structuring elements of growing size

- $B =$ ■

X $\delta_{5B}(X)$ $\delta_{10B}(X)$ $\delta_{15B}(X)$

Dilatation and Minkowski 's addition

- The dilatation is equivalent to the set addition by the transposed element.

$$\delta_B(X) = X \oplus \check{B} = \bigcup_{b \in B} X_{-b}$$

- Demonstration : Let $z \in \bigcup_{b \in B} X_{-b}$...(to do...)

Set erosion and dilatation's properties

Duality

Erosion and Dilatation are related. We obtain the same result by eroding X or by dilating its complementary and taking the complementary of the resulting set.
We say that Erosion and Dilatation are two dual operations according to the complementation.

$$\begin{cases} \mathcal{E}_B(X)) & = & \mathcal{C}_E(\delta_B(\mathcal{C}_E(X))) \\ \delta_B(X)) & = & \mathcal{C}_E(\mathcal{E}_B(\mathcal{C}_E(X))) \end{cases}$$

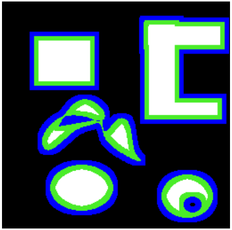
X $\delta_{5B}(X)$ $\mathcal{E}_{5B}(\mathcal{C}_E(X))$

Extensibility

- Dilatation is an extensive transformation while erosion is counter-extensive.

$$\mathcal{E}_B(X) \subset X \subset \delta_B(X)$$

- $\mathcal{E}_B(X) = \square$
- $X = \square + \blacksquare$
- $\delta_B(X) = \square + \blacksquare + \blacksquare$



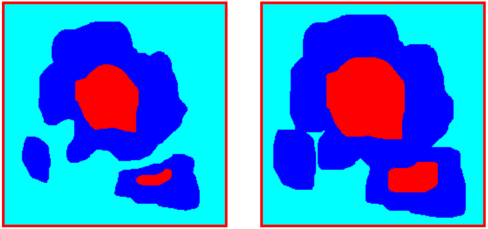
Growth

- Dilatation and erosion are increasing operators

$$\begin{cases} X \subset Y \Rightarrow \delta_B(X) \subset \delta_B(Y) \\ X \subset Y \Rightarrow \mathcal{E}_B(X) \subset \mathcal{E}_B(Y) \end{cases}$$

- Erosion is **decreasing** according to the size of the structuring element.

$$B \subset B' \Rightarrow \mathcal{E}_{B'}(X) \subset \mathcal{E}_B(X)$$



■ ⊂ ■ $\qquad$ $\delta_B(■) \subset \delta_B(■)$

Composition

- Dilatation with a structuring element of size n is equal to n dilatations with a struturing element of size 1. (idem for erosion)

$$\delta_{nB}(X) = \underbrace{\delta_{1B} \circ \cdots \circ \delta_{1B}}_{nfois}(X)$$

  nB scaling of B by a factor n.

-  Useful for an hardware implementation when the size of the struturing element is bounded by hardware constraints : A larger structuring element may be obtained by using a cascade of operators.

Union, Intersection

- Dilatation commutes with union :

$$\delta_B(X \cup Y) = \delta_B(X) \cup \delta_B(Y)$$

- Erosion commutes with intersection :

$$\mathcal{E}_B(X \cap Y) = \mathcal{E}_B(X) \cap \mathcal{E}_B(Y)$$

- Moreover :

$$\left\{ \begin{array}{lcl} \delta_{B_1 \cup B_2}(X) & = & \delta_{B_1}(X) \cup \delta_{B_2}(X) \\ \mathcal{E}_{B_1 \cup B_2}(X) & = & \delta_{B_1}(X) \cap \mathcal{E}_{B_2}(X) \end{array} \right.$$

- But :

$$\left\{ \begin{array}{lcl} \delta_{B_1 \cap B_2}(X) & \subset & \delta_{B_1}(X) \cap \delta_{B_2}(X) \\ \mathcal{E}_{B_1 \cap B_2}(X) & \supset & \mathcal{E}_{B_1}(X) \cup \mathcal{E}_{B_2}(X) \\ \mathcal{E}_B(X \cup Y) & \supset & \mathcal{E}_B(X) \cup \mathcal{E}_B(X) \end{array} \right.$$

Composition

- Let B_1 and B_2 be two symmetric structuring elements according to their origins. We have :

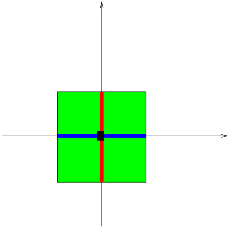
$$\left\{ \begin{array}{lcl} \delta_{\delta_{B_1}(B_2)}(X) & = & \delta_{B_1}(\delta_{B_2}(X)) \\ \mathcal{E}_{\mathcal{E}_{B_1}(B_2)}(X) & = & \mathcal{E}_{B_1}(\mathcal{E}_{B_2}(X)) \end{array} \right.$$

- Demonstration (for dilatation) :

$$\begin{array}{lcl} \delta_{\delta_{B_1}(B_2)}(X) & = & X \oplus (B_1 \oplus \breve{B}_2)\breve{} \\ & = & X \oplus (B_1 \oplus B_2)\breve{} \\ & = & X \oplus (B_1 \oplus B_2) \\ & = & (X \oplus B_1) \oplus B_2 \\ & = & (X \oplus \breve{B}_1) \oplus \breve{B}_2 \\ & = & \delta_{B_1}(\delta_{B_2}(X)) \end{array}$$

: key point

Composition : Illustration



- B_1 : [red line]
- B_2 : [blue line]
- $\delta_{B_1}(B_2)$: [green square]

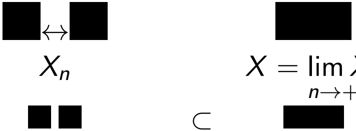
We pass from a $\mathcal{O}(n^2)$ complexity to a $\mathcal{O}(2n)$ one.

- Erosion is semi-continuous superiorly

$$\forall B \ \overline{\lim_{n \to +\infty}} \mathcal{E}_B(X_n) \subset \mathcal{E}_B(\lim_{n \to +\infty} X_n)$$

- Demonstration : (to do)
- Dilatation is a continuous operator.

- Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of 2 squares separated by a distance of $\frac{1}{n}$.
- $B =$ ■



X_n $\quad\quad$ $X = \lim_{n\to+\infty} X_n$

$\overline{\lim}_{n\to+\infty}\, \mathcal{E}_B(X_n) \subset \mathcal{E}_B(X)$

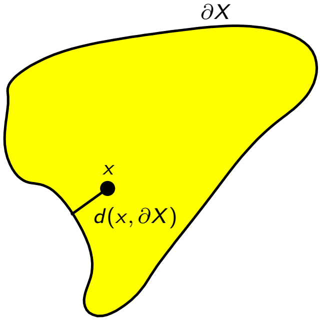
- Dilatation removes small holes, small isthmus and makes objects bigger.
- Erosion removes small objects, small isthmus, and shrinks objects.
- Dilatation and erosion are not topological transformations.
- If X is connected $\delta_B(X)$ is connected
- Dilatation and erosion are non reversible operations
- Dilatation and erosion are dual but not inverse operations :

$$\begin{array}{ccccc} & & \textit{dilatation} & \\ & X & \longrightarrow & \delta_B(X) \\ \text{complémentation} & \updownarrow & & \updownarrow \\ & \mathcal{C}_E(X) & \longrightarrow & \mathcal{E}_B(\mathcal{C}_E(X)) \end{array}$$

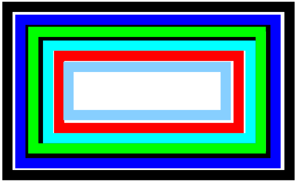
Calculs de Distances

- Let $x \in X$

$$d(x, \partial X) = d(x, \mathcal{C}_E(X)) = \inf_{y \in \mathcal{C}_E(X)} d(x, y) = \inf_{y \in \partial X} d(x, y)$$



Distance and slices

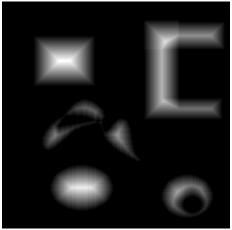


- ■ : dist=0
- ■ : dist=1
- ■ : dist=2
- ■ : dist=3
- ■ : dist=4...

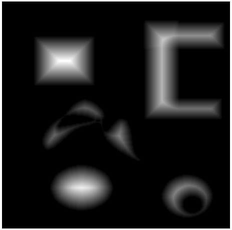
Distance by erosion : Algorithm

- bin1=initial image
- Initialization of an empty image grey0
- copy of bin1 in grey0 ($f(x) = 1$ if $x \in X$, 0 otherwise)
- While bin1 not empty do
  - bin1$\leftarrow \mathcal{E}_{1B}(bin1)$
  - grey0 $\leftarrow$ grey0 + bin1
- End While
- grey0 = distance function + 1

Shapes



Erosion based distance

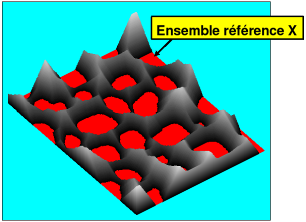


Euclidean distance

- Distance by erosion is only an approximation of the Euclidean distance.

External distance

- Distance to a set obtained by dilatation



Extension to the distance between two sets

$$d(X,Y) = \inf_{(x,y)\in X\times Y} d(x,y)$$

- bin_0 = binary image containing X
- extraction of the connected component X_i which is copied into the image bin_1
- $bin_2 = bin_0/bin_1$ (we put in bin_2all others connected components)
- Creation of an empty image bin3
- d = 0 (initialisation of the value of the distance between X_i and (X/X_i)
- While bin_3 is empty
- do
  - $bin_1 \leftarrow \delta_{1B}(bin_1)$
  - d = d + 1
  - $bin_3 = bin_1 \cap bin_2$
- End while

Erosion and dilatation of functions

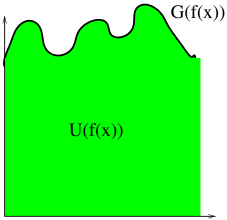
- Pb : Erosion and dilatation have been defined only in the framework of set theory. What to do with $f : \mathbb{R}^2 \rightarrow \mathbb{R}$?

- Idea : Consider a function as a set
  - Graph of a function :

$$G(f(x)) = \{(x,t) \mid t = f(x)\}$$

  - Shadow of a function :

$$U(f(x)) = \{(x,t) \mid t \leq f(x)\}$$



Type of structuring element

- $U(f(x))$ is composed of couples (x, t) with $t = f(x)$.
- The structuring element is composed of couples $(x, b(x))$ with x belonging to a bounded support.

$$B = \{(x, t) \,|\, x \in B' \text{ and } t \leq b(x)\}$$

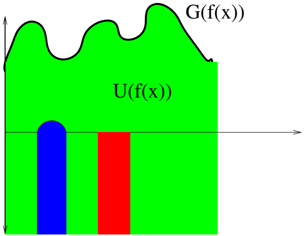
- b fulfils the following conditions :

$$b(x) = \begin{cases} x \in B' & \Rightarrow & b(x) & \neq & \pm\infty \\ x \notin B' & \Rightarrow & b(x) & = & -\infty \end{cases}$$

- We distinguish the flat structuring element defined by :

$$b(x) = \begin{cases} x \in B' & \Rightarrow & b(x) & = & 0 \\ x \notin B' & \Rightarrow & b(x) & = & -\infty \end{cases}$$

Examples of structuring elements



- Non flat structuring elements are called volumic structuring elements.

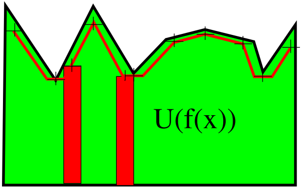
Erosion of a function

- The erosion of a function is defined as the set-erosion of its shadow.

$$\mathcal{E}_B(U(f(x))) = \{(x,t) \mid B_{(x,t)} \subset U(f(x))\}$$

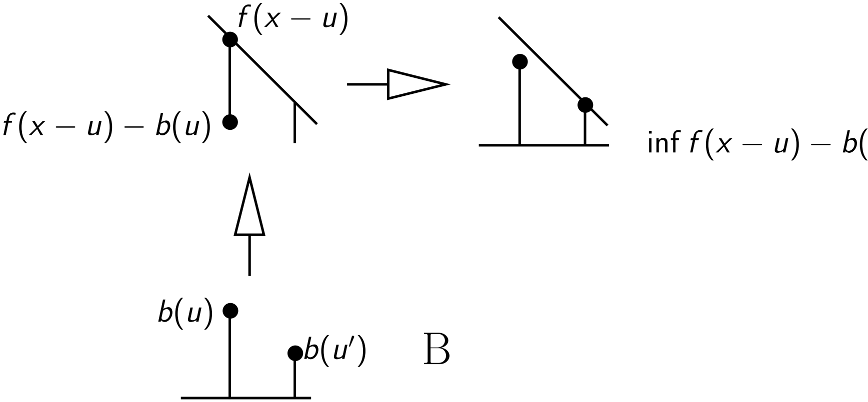
- Question : Given x what is the maximal value of t such that $B_{(x,t)} \subset U(f(x))$?
- Answer : Using a flat structuring element : the smallest value of f on its support B'.

$$\mathcal{E}_B(f)(x) = \inf_{u \in B'} f(x-u)$$



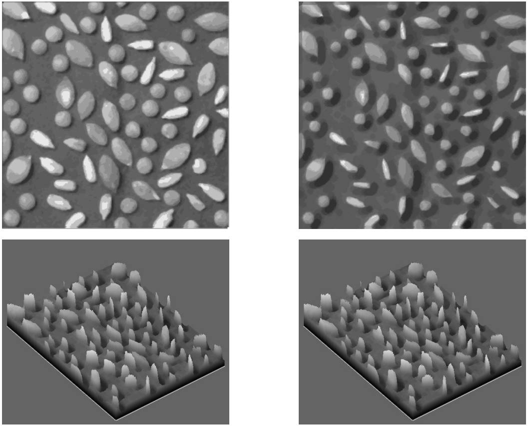
- Heights $b(u)$ must be in $U(f(x))$

$$\mathcal{E}_B(f)(x) = \inf_{u \in B'} f(x-u) - b(u)$$



Example of erosion (1)

Original — Eroded (disc of radius 3)



The eroded image is darker, its pics shrink or disapear.

Examples of erosion (2)

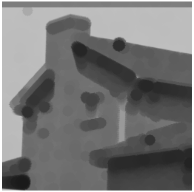
- Erosion with a flat structuring element having a circular support

Original

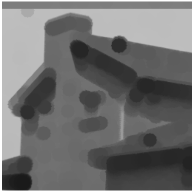
$\mathcal{E}_{2B}(I)$

$\mathcal{E}_{3B}(I)$

$\mathcal{E}_{5B}(I)$



$\mathcal{E}_{7B}(I)$



$\mathcal{E}_{9B}(I)$

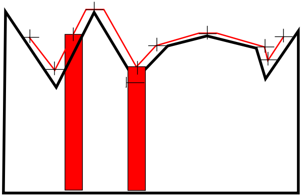
Dilatation of a function

- Defined the same way using the function's shadow

$$\delta_B(U(f(x))) = \{(x,t) \,|\, B_{(x,t)} \cap U(f(x)) \neq \emptyset\}$$

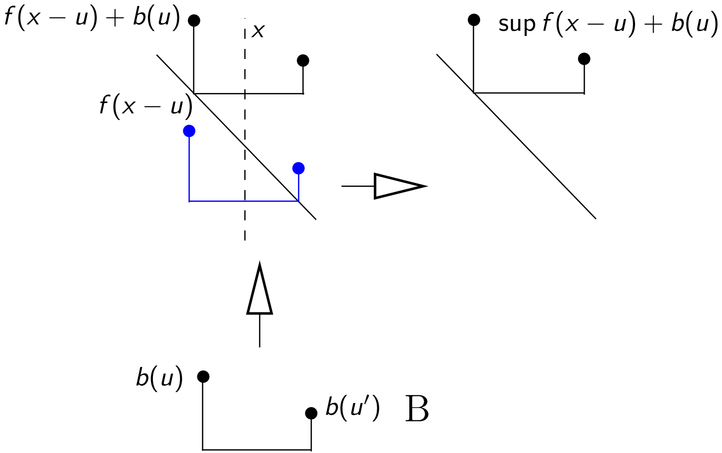
- Question : given x what is the maximal value of t such that $B_{(x,t)} \cap U(f(x)) \neq \emptyset$?
- Answer : In the case of a flat structuring element : The maximal value of f on its support B' :

$$\delta_B(f)(x) = \sup_{u \in B'} f(x-u)$$



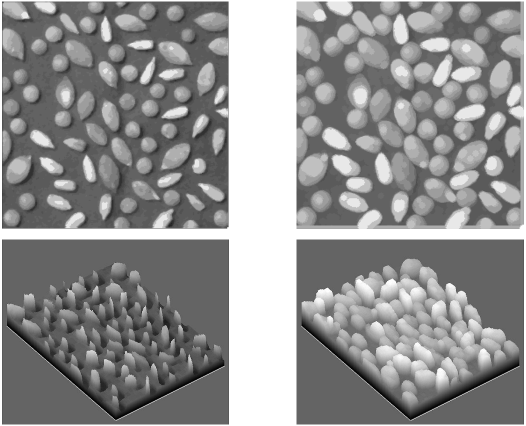
- De façon symétrique à l'érosion :

$$\delta_B(f)(x) = \sup_{u \in B'} f(x-u) + b(u)$$



Example of dilatation (1)

Original dilated (disc of radius 3)



The dilated image is more bright.
Narrow valleys disappear.

Example of dilatation (2)

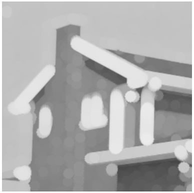
- Dilatation with a circular, flat structuring element

Original

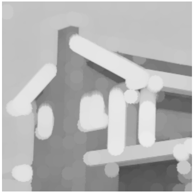
$\delta_{2B}(I)$

$\delta_{3B}(I)$

$\delta_{5B}(I)$



$\delta_{7B}(I)$

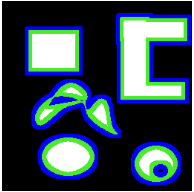


$\delta_{9B}(I)$

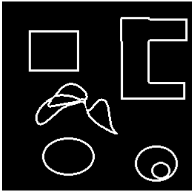
Morphological Residues

- Within set morphology a residue corresponds to :
  - The symmetric difference between the initial set X and its transformation $\psi(X)$.
  - The symmetric difference between two transformations $\psi_1(X)$ et $\psi_2(X)$ of a set .
- Within functional morphology :
  - Arithmetic difference between $f(x)$ and $\psi(f(x))$,
  - Arithmetic difference between $\psi_1(f(x))$ and $\psi_2(f(x))$.
- When transformations ψ, ψ_i are defined as erosion or dilatation we speak of morphological gradients.

X



$\mathcal{E}_B(X) = \square, X = \square + \blacksquare$



$X \Delta \mathcal{E}_B(X)$

Morphological gradients

Morphological Gradients

- Three types of gradients :
  - The symmetric morphological gradient (from Beucher) :

    $\nabla_B(X) = \delta_B(X)\Delta\mathcal{E}_B(X)$ and $\nabla_B(f(x)) = \delta_B(f(x)) - \mathcal{E}_B(f(x))$

  - Internal (or by erosion) morphological gradient :

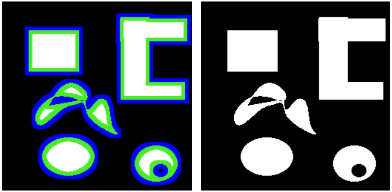
    $\nabla_B^-(X) = X\Delta\mathcal{E}_B(X)$ and $\nabla_B^-(f(x)) = f(x) - \mathcal{E}_B(f(x))$

  - External (or by dilatation) morphological gradient :

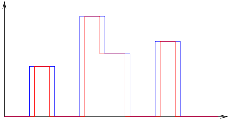
    $\nabla_B^+(X) = \delta_B(X)\Delta X$ and $\nabla_B^+(f(x)) = \delta_B(f(x)) - f(x)$

Morphological set gradients

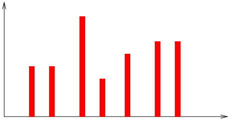
- Sets :
  - $\mathcal{E}_B(X) = \square$
  - $X = \square + \textcolor{green}{\blacksquare}$
  - $\delta_B(X) = \square + \textcolor{green}{\blacksquare} + \textcolor{blue}{\blacksquare}$
- Gradients :
  - $\nabla_B(X) = \textcolor{green}{\blacksquare} + \textcolor{blue}{\blacksquare}$
  - $\nabla_B^+(X) = \textcolor{blue}{\blacksquare}$
  - $\nabla_B^-(X) = \textcolor{green}{\blacksquare}$



Internal morphological gradient



- $f(x)$: blue line
- $\mathcal{E}_B(f(x))$: red line
- $\nabla_B^-(f(x))$: red bars

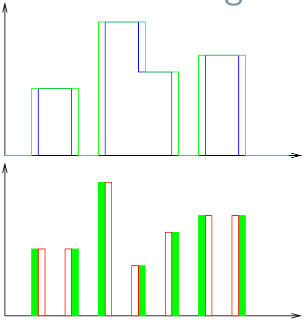


I

$\mathcal{E}_B(I)$

$\nabla_B^-(I)$

External morphological functional gradient



- $f(x)$:
- $\delta_B(f(x))$:
- $\nabla_B^+(f(x))$:
- $\nabla_B^-(f(x))$:

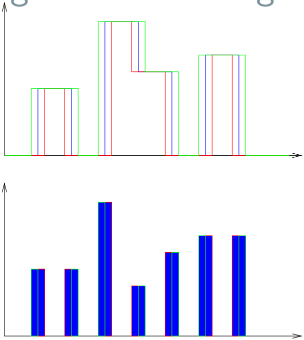
I

$\delta_B(I)$

$\nabla_B^-(I)$

Symmetric morphological functional gradient

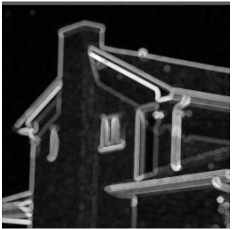
- $f(x)$:
- $\delta_B(f(x))$:
- $\mathcal{E}_B(f(x))$:
- $\nabla_B(f(x))$:
- $\nabla_B^-(f(x))$:
- $\nabla_B^+(f(x))$:



I

$\delta_B(I)$

$\mathcal{E}_B(I)$



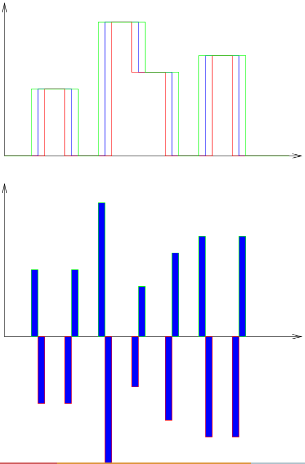
$\nabla_B^-(I)$

Morphological Laplacian

- The morphological Laplacian is defined as the residue of the external and internal gradients.

$$\mathcal{L}_B(f) = \nabla^+_B(f) - \nabla^-_B(f)$$

- $f(x)$:
- $\delta_B(f(x))$:
- $\mathcal{E}_B(f(x))$:
- $\mathcal{L}_B(f(x))$:
- $\nabla^-_B(f(x))$:
- $\nabla^+_B(f(x))$:

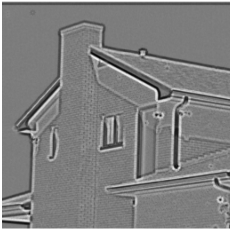


Morphological Laplacian : Example

I

$\nabla_B^-(I)$

$\nabla_B^+(I)$



$\mathcal{L}_B(I)$